Metric Spaces and Topology
Lecture 26

However, for first ctbl spaces, compactness $\Rightarrow$ sequential compactness. This and the following other properties listed below are left for homework.

Properties of 1^{st}-ctbl spaces. Let X be a 1^{st}-ctbl top. space.

(a) Each pt. $x \in X$ admits a ctbl decreasing neighb. basis, i.e. a basis $(B_n)_{n\in\mathbb{N}}$ s.t. $B_n \supseteq B_{n+1}$ $\forall n \in \mathbb{N}$.

(b) For any $Y \subseteq X$ and $x \in X$, $x \in \overline{Y} \iff \exists\, (y_n) \subseteq Y$ converging to x.

(c) If $(x_n) \subseteq X$ doesn't have a convergent subsequence, then the set $\{x_n : n \in \mathbb{N}\}$ is closed.

(d) If X is compact, then it is sequentially compact.

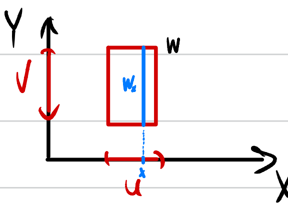
We now prove:

Tychonoff's Theorem (AC). Any product (possibly unctbl) of compact top spaces is compact (in the product top).

To motivate the proof, let's try proving that the product $X \times Y$ of compact spaces X, Y is compact. Let $\mathcal{W}$ be an open cover of $X \times Y$. Firstly, we may assume WLOG that $\mathcal{W}$ consists of open rectangles $U \times V$, where $U \subseteq X$, $V \subseteq Y$ open.
Then $\mathcal{W}_X := \{U \subseteq X : \exists V \subseteq Y \text{ s.t. } U \times V \in \mathcal{W}\}$,

$\mathcal{W}_Y := \{V \subseteq X : \exists U \subseteq X \text{ s.t. } U \times V \in \mathcal{W}\}$.

It follows that $\mathcal{W}_X$ and $\mathcal{W}_Y$ are open covers of X and Y, resp., so $\exists$ finite subcovers $\mathcal{W}_X' \subseteq \mathcal{W}_X$ and $\mathcal{W}_Y' \subseteq \mathcal{W}_Y$.



It's clear that $\mathcal{W}_X' \times \mathcal{W}_Y'$ is a finite cover of $X \times Y$, but it may not be contained in $\mathcal{W}$.

Another idea (Hayk Karapetyan): For each pt. $x \in X$, $\mathcal{W}$ is an open cover of $\{x\} \times Y$, and $\{x\} \times Y$ is compact as it is homeomorphic to Y. Thus, $\exists$ finite subcover $\mathcal{W}_x \subseteq \mathcal{W}$ of $\{x\} \times Y$. Let $U_x := \bigcap_{W \in \mathcal{W}_x} \mathrm{proj}_X W$. Then $\{U_x : x \in X\}$ is an open cover of X, so $\exists$ finite subcover $U_{x_1}, U_{x_2}, \dots, U_{x_n}$. Then $\bigcup_{i=1}^{n} \mathcal{W}_{x_i}$ is a cover of $X \times Y$, and a subset of $\mathcal{W}$. This proof also works for finite products $X_1 \times X_2 \times \dots \times X_n$ since the role of Y will be played by X_n and the role of X will be played by $X_1 \times X_2 \times \dots \times X_{n-1}$, which is compact by induction.

However for infinite products this won't work.

First a remark on covers and refinements.

Def. For a cover $\mathcal{U}$, a cover $\mathcal{V}$ is called a *refinement* of $\mathcal{U}$ (or $\mathcal{U}$ is called a *coarsening* of $\mathcal{V}$) if $\forall\, V \in \mathcal{V}$ $\exists\, U \in \mathcal{U}$ s.t. $V \subseteq U$.

Obs. If a cover $\mathcal{V}$ refines $\mathcal{U}$ and $\mathcal{V}$ admits a finite sub-cover, then so does $\mathcal{U}$.

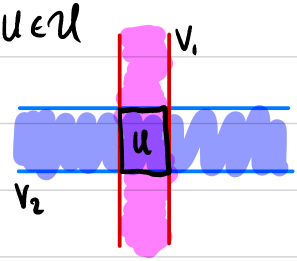
Cor. A top sp. X is compact $\iff$ every basic open cover has a finite subcover, more precisely, for some basis $\mathcal{B}$, every cover $\mathcal{U} \subseteq \mathcal{B}$ of X has a finite subcover.

Proof. $\Leftarrow$. Given any open cover $\mathcal{U}$, there is a refinement $\mathcal{U}_{\mathcal{B}} \subseteq \mathcal{B}$, indeed, take $\mathcal{U}_{\mathcal{B}} := \{B \in \mathcal{B} : \exists\, U \in \mathcal{U}\ B \subseteq U\}$. Since $\mathcal{U}_{\mathcal{B}}$ has a fin. subcover, so does $\mathcal{U}$ by the Obs. above. $\square$

Note that in the case $X \times Y$ above, if we could assume that $\mathcal{W}$ consisted of cylindrical sets $U \times X$ and $X \times V$, then either the U's would form an cover of X or the V's would form a cover of Y (otherwise $\mathcal{W}$ isn't a cover). Either way $\mathcal{W}$ admits a finite subcover.

Alexander's prebasis lemma (AC). A top. sp. X is compact $\Leftrightarrow$ any prebasic open cover has a finite subcover, more precisely, for some prebasis $\mathcal{P}$, any cover $\mathcal{U} \subseteq \mathcal{P}$ has a finite subcover.

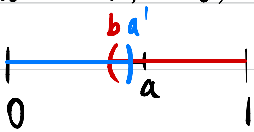
Proof $\Leftarrow$. Fix a prebasis $\mathcal{P}$ and let $\mathcal{B}_{\mathcal{P}}$ be the basis generated by $\mathcal{P}$, i.e. $\mathcal{B}_{\mathcal{P}}$ is the collection of finite intersections of sets in $\mathcal{P}$. It's enough to show that any cover $\mathcal{U} \subseteq \mathcal{B}_{\mathcal{P}}$ has a finite subcover. Suppose $\exists$ such $\mathcal{U} \subseteq \mathcal{B}_{\mathcal{P}}$ with no finite subcover. By Zorn's lemma, $\boxed{\text{HW}}$ let $\mathcal{U} \subseteq \mathcal{B}_{\mathcal{P}}$ be an inclusion-maximal cover with no finite subcover. Then $\mathcal{U} \cap \mathcal{P}$ is not a cover of X,



so $\exists\, U \in \mathcal{U}$ s.t. $\exists\, V \in \mathcal{U} \cap \mathcal{P}$ containing U. But $U = V_1 \cap V_2 \cap \ldots \cap V_n$ with $V_i \in \mathcal{P}$, so by the maximality of $\mathcal{U}$, for each i, the cover $\mathcal{U} \cup \{V_i\}$ has a finite subcover $\mathcal{U}_i \cup \{V_i\}$.

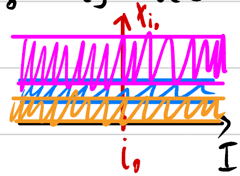
This $\mathcal{U}_i$ covers V_i^c, so $\bigcup_{i=1}^n \mathcal{U}_i$ covers $\bigcup_{i=1}^n V_i^c = \left(\bigcap_{i=1}^n V_i\right)^c = U^c$, so $\{U\} \cup \bigcup_{i=1}^n \mathcal{U}_i$ is a finite subcover of $\mathcal{U}$. □

Easy application. Let's reprove that $[0,1]$ is compact. The intervals of the form $[0,a)$ and $(b,1]$ form a prebasis, so take a cover $\mathcal{U}$ of $[0,1]$ with such intervals. Let $a := \sup\{a' \in [0,1] : [0,a') \in \mathcal{U}\}$. Then $a \in [0,1]$ so some $U \in \mathcal{U}$ has to cover it and by def, U can't be of the form $[0,a')$ because then $a' > a$, contradicting the def of a. Thus, $U = (b,1]$ and $b < a$. But then $\exists a'$ $b < a' < a$ s.t. $[0,a') \in \mathcal{U}$, so $\{[0,a'), (b,1]\}$ is a finite subcover of $\mathcal{U}$. □



Proof of Tychonoff's theorem. By Alexander's lemma, it's enough to take a cover $\mathcal{W}$ of $X := \prod_{i\in I} X_i$ with sets of the form $[i \mapsto U_i]$, where $i \in I$, $U_i \subseteq X_i$ open. We first note that $\exists i_0 \in I$ s.t. $\mathcal{W}_{i_0} := \{U \subseteq X_{i_0} : [i_0 \mapsto U] \in \mathcal{W}\}$ covers X_{i_0}; indeed, otherwise by AC $\exists$ $x := (x_i)_{i \in I}$ s.t. $x_i \in X_i \setminus \bigcup \mathcal{W}_i$, but then $x \notin \bigcup \mathcal{W}$ because if $x \in [i \mapsto U_i] \in \mathcal{W}$, then $U_i \in \mathcal{W}_i$ and $x_i \notin U_i$ so $x \notin [i \mapsto U_i]$. Since X_{i_0} is compact,

$\mathcal{W}_{i_0}$ has a finite subcover $\mathcal{W}'_{i_0}$ so $\{[i_0 \mapsto U] : U \in \mathcal{W}'_{i_0}\}$ is a subset of $\mathcal{W}$ and it covers X.



$\square$